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THE CHALLENGE OF ABUL WAFA

Jean Le MÉE

Introduction

Abul Wafa's full name was *Abū al-Wafā', Muḥammad ibn Muḥammad ibn Yaḥyā ibn Ismā'īl ibn al-'Abbās al-Būzjānī or Abū al-Wafā Būzhgānī.* He was born on the 10th June 940, and died on the 15th July 998. He was a Persian philosopher, mathematician and astronomer who worked in Baghdad. Among other things, he developed the field of trigonometry. In particular, he introduced the tangent function. He is also credited[1] with 'the feat of drawing all five Platonic solids, (Fig.1a), using only a straightedge and a pair of compasses at a fixed setting.' Such fixed compasses (known as 'rusty' compasses), adds Hersey[2], have been the tools of virtuoso geometrical draftsmanship in many periods.

We would like to show here, without laying claim to virtuosity eleven centuries after, that the deed can easily be accomplished. Indeed it is already outlined in Euclid's *Elements*. We first look at Euclid's method and then propose an approach based on the Maraldi angle.

1. Euclidean Approach

In proposition 18 of his Book XIII, *The Elements*, Euclid had already given a geometric construction for the edges of all five Platonic solids inscribed within a common circumsphere[3].

1 Hersey, G.L., (2000). *Architecture and Geometry in the Age of the Baroque*, Chicago UP, p. 88. Ibid

2 Ibid.

3 March, L., (1998). Architectonics in Humanism: *Essays on Number in Architecture*. Chichester, UK: Academy Ed. John Wiley& Sons Ltd. pp. 88-90.

The construction in this case with a set compass is straightforward except for the dodecahedron. With some additional steps, however, outlined at the end of this section, it can easily be achieved.

Abul Wafa (940-998)

The Euclidean approach limits itself to giving the lengths of the edges of each of the five Solids. The approach that we propose in the next section suggests also an actual construction of the solids.

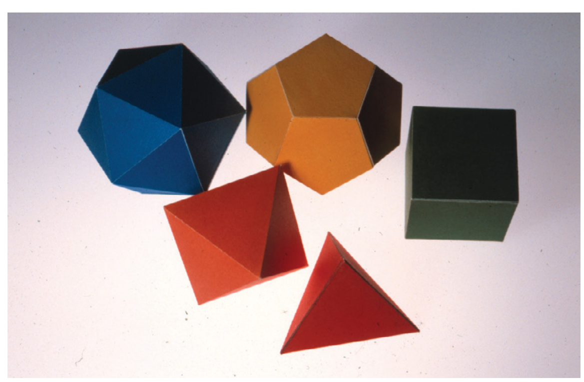

Figure 1a. The Five Platonic Solids.

Referring to figure 2, the circle of center O and diameter AB is a great circle of the sphere circumscribing the five Platonic solids. ABCD is a square built on AB.

Diagonals DB and AC meet at E. DO and CO cross AC and DB at X and Y respectively. CO cuts circumference of circle O at G. Draw triangle AGB. Through X, E, and Y draw perpendiculars to AB. Perpendicular through Y intersects circumference at F. Draw triangle AFB.

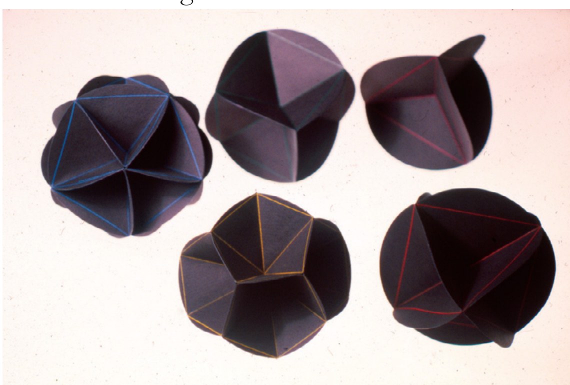

Figure 1b. The Five Platonic Solids and their Maraldi Angle.

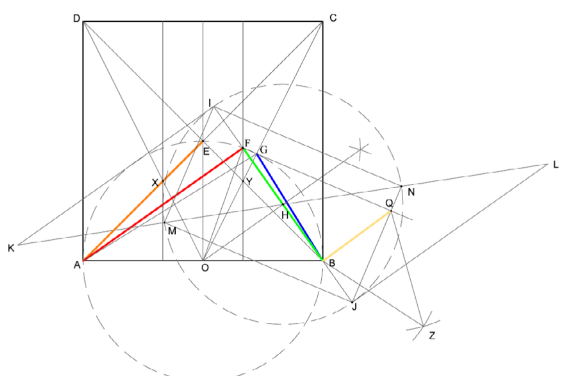

Figure 2. Euclidean Construction of Edges of Platonic solids.

Then the half-square diagonal (AE) is the length of the edge of the octahedron. FB is the length of the edge of the cube; AF is the length of the tetrahedron; and GB, the length of the edge of the icosahedron. Note that all these segments result from operations with a straight edge and a single compass opening. We leave the proof to the reader who may want to refer to Euclid himself with the help of Heath[4], Serlio[5], and Fowler[6].

Examination of the dodecahedron reveals that since its faces are pentagons and the diagonals of the pentagons form a cube of common circumsphere with the dodecahedron, the relation between cube edge and dodecahedron edge is in the Golden Ratio φ. Given the cube edge FB, the problem is to find a segment that would be equal to the short side of the Golden Rectangle where FB is the longer side.

Such a construction, however, can easily be achieved with a single compass setting:

- Extend FB both ways and draw its mediatrix intersecting FB at H.
- Draw circle centered at H with rusty setting.
- Build perpendiculars at the end I and J of diameter containing segment FB. This is most easily done by drawing the tangents to both circles O and H through I and J.
- Extend these perpendiculars by the length of one diameter on alternate sides and join the extremities K and L.
- KL cuts circle of center H at M and N.
- INJM is a Golden Rectangle and therefore IN = φNJ.
- Draw perpendicular to FB at B.
- Draw perpendicular from F to NJ.
- Both meet at Q.
- FQB, similar to IJN, therefore FB/BQ = IN/JN = φ.
- But since FB is edge of cube, then BQ is edge of dodecahedron.

AF, tetrahedron edge subtends angle AOF.
FB, cube " " " " FOB.
AE, octahedron " " " AOE.
GB, icosahedron " " " GOB.

By striking arcs BZ and QZ (with rusty compass setting) one forms angle BZQ.

All these angles are the Maraldi angles defined in the next section, showing the two methods to provide the same information, the first through geometric means alone, the second, through the means of trigonometric ratios.

2. Maraldian Approach

The method proposed here is based on a construction of the Maraldi angle for each of the five Platonic solids. The Maraldi angle, also called internal angle, is the angle formed by any pair of consecutive radii joining the center of the circumscribing sphere to both ends of a given

4 Heath, T. L., (1956). *The Thirteen Books of Euclid's Elements*, 2nd edition. New York: Dover.

5 Hart, V. and Hicks, P., trans. (1996). *Sebastiano Serlio: On Architecture*. New Haven: Yale University Press.

6 Fowler, D., (1999). *The Mathematics of Plato's Academy: A New Reconstruction*. 2nd edition. New York: Oxford University Press (Chapter 5 and Appendix 10.1).

edge, i.e. to the corresponding vertices of the regular solids being circumscribed.

It is easy to demonstrate that the trigonometry of the Maraldi angles for the five Platonic solids is given by the ratios of the first three integers - something that would have pleased Plato and that Abul Wafa might have known. Thus for the cube, C is the internal or Maraldi angle with cos Ci = X/R = 1/3 (Figs. 3, 4, & 5), where 'a' is the edge of the cube. CA=R, the half-diagonal of the cube or the radius of the circumsphere, X=CH is obtained by dropping the perpendicular AH from A on CB. Figure 6 shows the construction of the internal angles for the five Platonic solids.

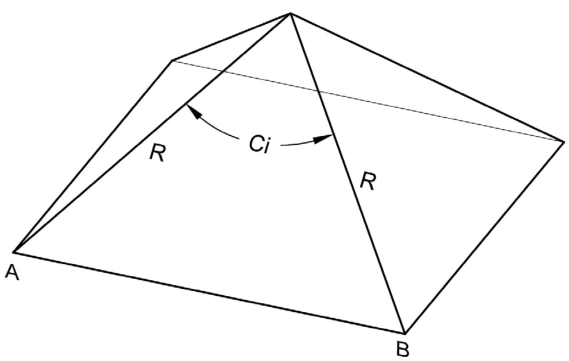

Figure 3. Cube Constitutive Pyramid.

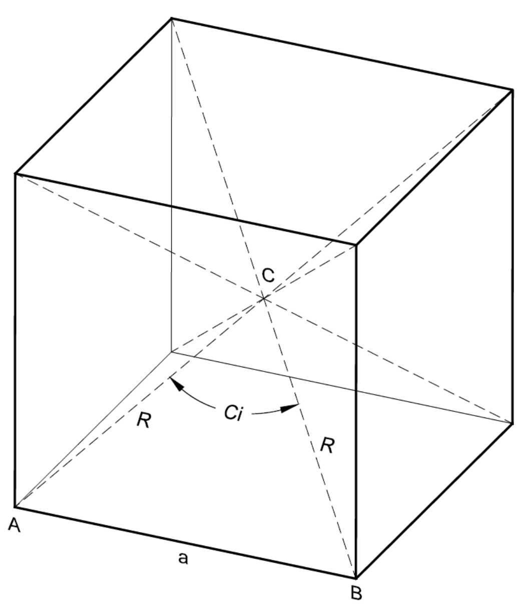

Figure 4. Cube.

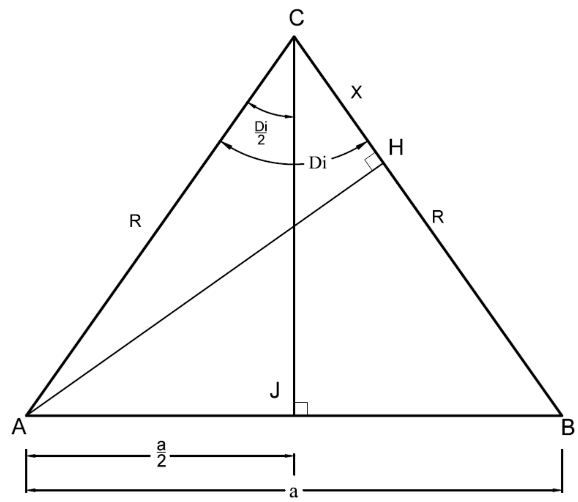

Figure 5. Face of Pyramid.

The cube is seen as being formed of six pyramids with square bases of side 'a' and a common apex. The sloping edges of each pyramid are radii of the circumscribed sphere. Each such pyramid can therefore be constructed a shown on figures 8a and 8b.

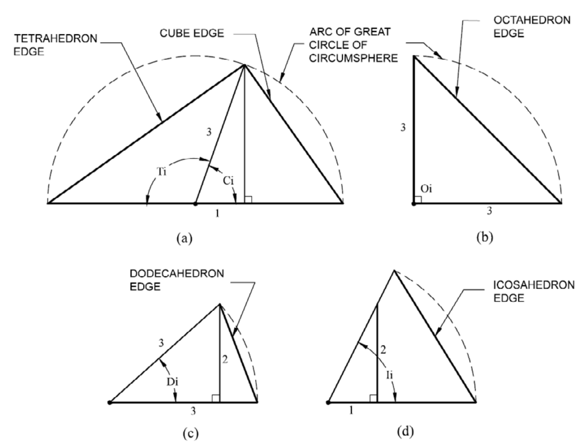

Figure 6.

Similarly, the tetrahedron is constituted of four pyramids with triangular bases and a Maraldi angle with cos Ti = -1/3 (Fig. 7a and 7b).

For the octahedron, we have eight pyramids with triangular bases and a Maraldi angle with sin Oi = 1 (Figs. 9a and 9b).

The dodecahedron has twelve pyramids with pentagonal bases and a Maraldi angle with sin Di = 2/3 (Figs. 10a and 10b).

Finally, the icosahedron will have twenty pyramids with a triangular basis and a Maraldi angle with tan Ii = 2 (Figs. 11a and 11b).

We can therefore construct the Maraldi angle and the edge typical of each of the Platonic solids as shown on figure 12. At this point we have used the adjustable compass to demonstrate the principle. We now show that this figure can be drawn with a 'rusty' compass.

We begin by tracing a line xy with the straightedge (Fig. 13). Setting our compass opening at R, radius of the circumsphere common to all the regular forms, we then proceed by drawing four intersecting circles whose centers O1,O2,O3,O4 are on line xy and the circumferences of their neighbors as shown on figure 13. The three vesicas determine both a square ABCD of side equal to the circle diameter and the median EF of the square to which we add diagonals AC and BD and square EHFG.

Now, draw IJ (it passes through O2) and join AO2 and DO2 cutting GE and GF at K and L respectively. Join KL cutting XY at M. Call center point of square ABCD, O5.

Then

MO5 = 2/3 GO5

or MO5 = 2/3 R

so that if we (arbitrarily) set R=3, MO5 = 2. This is easily established through similarity of triangles and is a standard construction for the harmonic series.

Now, repeat a similar construction along EF, i.e., draw NP cutting EF at Q. Then join G to Q and H to Q, cutting DB and AC at S and T respectively. ST cuts EF at U.

Then O5U = 1/3 O5F

or O5U = 1/3 R

again with R=3, O5U =1.

Now with the same 'rusty' compass opening (Fig. 14), draw circle centered at O5. It is inscribed within square ABCD. Extend ST on both sides so that it cuts circle O5 at V1.

On the left, it passes through L; O5 L extended cuts the circumference at V2 and KL extension will cut it at V3.

Join FV1, FV2, FV3, EV1, EG.

These are the edges respectively of: The cube, the icosahedron, the dodecahedron, the tetrahedron, and the octahedron. Each of them subtends the central angle that is the Maraldi angle of the corresponding polyhedron. The triangle formed by the edges of each of the solids indicated and the radii ending at the extremities of the edges constitute the faces of the pyramids.

To complete the challenge it remains to construct each of the solids. In our case, the problem is to draw the development of each of the pyramids making up each solid. If we desire to construct Platonic solids that can be inscribed in a common sphere, patterns can be cut out to build all the pyramids from the triangles.

If we want to draw the face of the pyramid at any scale, it is just sufficient to draw the bisector of each of the Maraldi angles (an easy task with a rusty compass) and draw on this a perpendicular at any desired point marking the height of the face of the pyramid.

The Abul Wafa challenge has therefore been met.

Comparing the two methods we see that they are roughly equivalent in terms of number of steps to get the data, namely the length of the edge and the corresponding Malardi angle of each of the five Platonic solids. If the first method seems to imply less circles, it is because the construction of square ABCD or that of the perpendiculars through X, E, Y, F, B have not been included.

For further applications of the Maraldi angles to regular stellated forms as well as to other topics such as Pythagorean triples, gnomic golden rectangle series, the golden and the exponential spirals, the tuning of the monochord and other things, the reader is referred to:

http://www.lemeestudies.com/Misc/14.millennium_sphere/products/AdQuadratum.pdf

Figure 7a, left; 7b, right.

Figure 8a, left; 8b, right.

Figure 9a, left; 9b, right.

Figure 10a, left; 10b, right.

Figure 11a., left;11b, right.

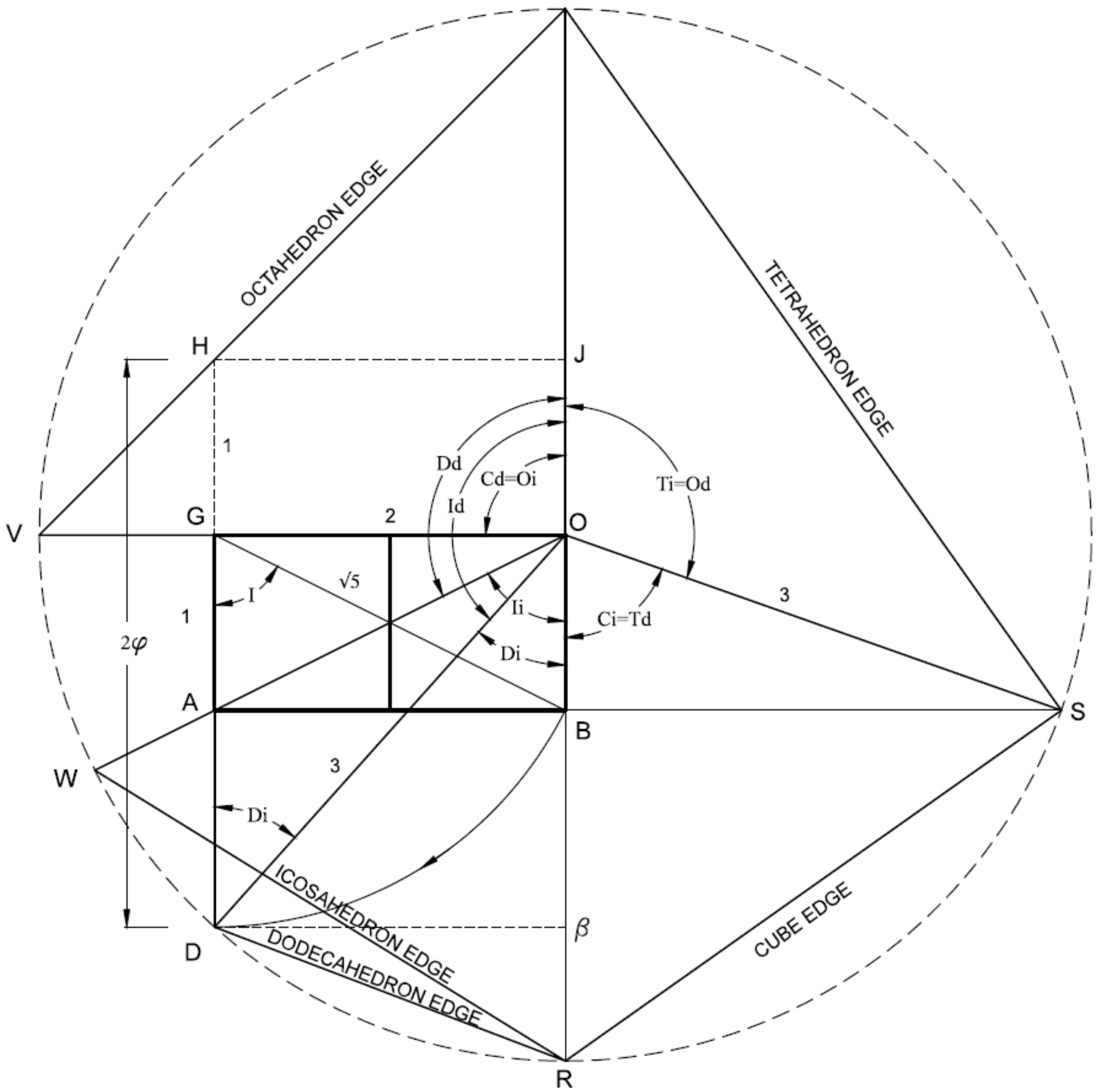

Figure 12. Construction of the Platonic Solid Edges.

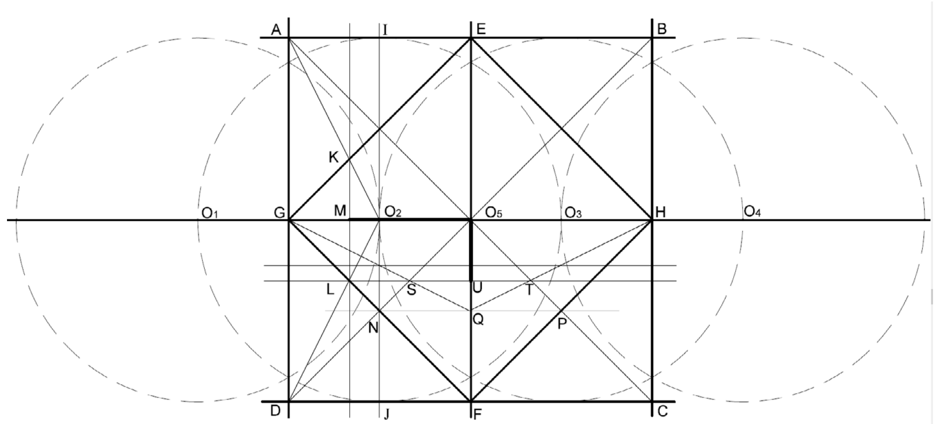

Figure 13. First Step Rusty Compass Construction of Platonic Solid Edges.

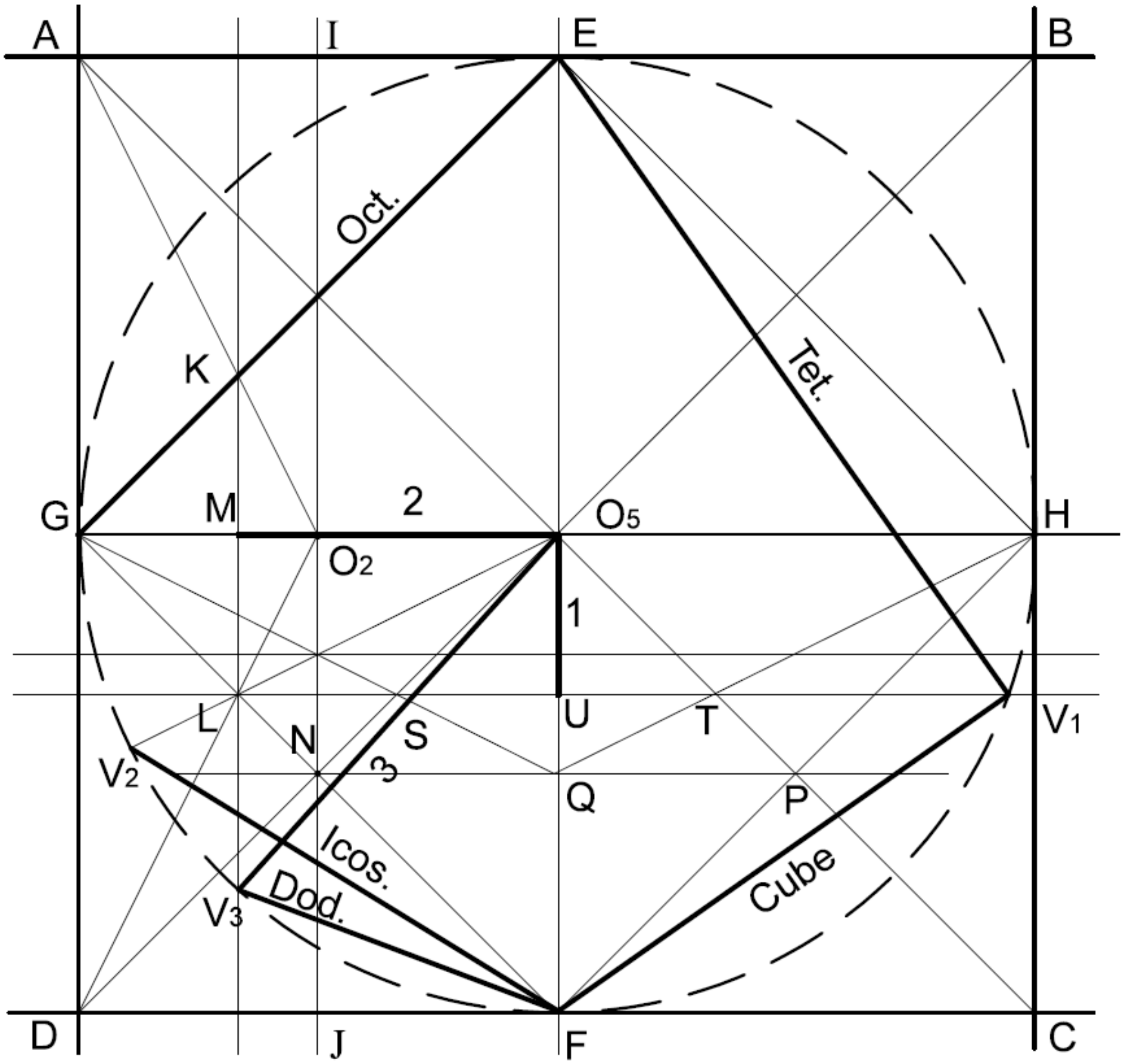

Fig. 14. Completion of the Construction of the Platonic Solid Edges.

In memoriam

Like with many such ‘chance’ meetings, whose significance becomes clear only much later, I came across Ernest McClain on Broadway, at the then Weiser bookshop, famous for its esoterica. This was in 1974, I believe, when I was working on the translation of some Hymns from the Rig Veda. I was foraging in a dark corner of the shop, looking for some references, when a man who had been floating about the place came to me and asked if I knew where he could find something on the Rig Veda! ‘Well, that’s precisely what I’m doing here,’ I said. There was an instant connection and for the next few years we met frequently over dinner, together with my wife Katharine and his wife Augusta, and had animated conversations related to some aspects of Vedic Philology, mathematics, meter, or philosophy and sundry topics. When, in retirement, he moved to New England, where we visited him once, the exchanges, naturally, became less frequent but in those pre-Internet days, we kept in touch by correspondence. Though I cannot say I followed all his developments on tuning theory as it applied to Plato, the Rig Veda, or the other areas of scholarship he probed, nevertheless, I must say that I was always impressed by the rigor of his analysis, the depth and clarity of his thought on any topic to which he directed his attention.

Above all, however, it is his generosity of spirit, his indefatigable will to explain what he conceived, and the communicative enthusiasm he brought to any exchange with fellow scholars that will remain imprinted in my memory.

His intellectual life has been a brilliant display worthy of a Fourth of July of the mind. All who had the privilege of knowing him may be grateful for it and those who will meet him, in the future, through his writings will glean there examples of a splendid scholarship.

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Acknowledgements

Special thanks to Dr. Marcin Balicki for handling the digitalization of the figures with his usual mathematical rigor and artistic sense and to Astrid Fitzgerald for her exquisite models of the Maraldian view of the Platonic forms (Figures 7a – 11a).